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(Residential Autonomous College under University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2015

FIRST YEAR

MATHEMATICS

Date : 26/5/2015

Time : 11 am – 1 pm

Paper : IX

Full Marks : 50

Answer **any five** questions : [5×10]

1. a) Let f be analytic within and on a simple closed curve γ and $\alpha \in \operatorname{int}\gamma$. Prove that $f(\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{(z-\alpha)}dz$, the integral being taken in the positive sense. Hence find the value of $\displaystyle\int_{|z|=2} \dfrac{\cos z}{z(z-3)}dz$. [5+2]

   b) Show that an entire function having its real part non-positive in $\mathbb{C}$ is a constant function. [3]

2. a) Prove that if an entire function f is bounded on $\mathbb{C}$, then f is a constant function. Hence deduce that, if $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ is a polynomial of degree $n \geq 1$, then the equation $f(z) = 0$ has at least one root in $\mathbb{C}$. [3+4]

   b) Evaluate the integral $\displaystyle\int_\gamma z^2 dz$, where γ is along x-axis from 0 to 2 and then along the line parallel to y-axis from 2 to 2 + 3i. [3]

3. a) Given the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ ... (1), let $R = \dfrac{1}{\rho}$ where $\rho = \limsup_{n\to\infty} |a_n|^{1/n}$ and $\gamma : |z - z_0| = R$. If $0 < R < \infty$, prove that the series (1) is absolutely convergent for all $z \in \operatorname{Int}\gamma$ and divergent for all $z \in \operatorname{Ext}\gamma$. [4]

   b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{n^n}{n!} z^n$. [3]

   c) Show that the power series $z - \dfrac{1}{2}z^2 + \dfrac{1}{3}z^3 - \ldots$ may be continued analytically to a wider region by means of the series $\log 2 - \dfrac{1-z}{2} - \dfrac{(1-z)^2}{2.2^2} - \dfrac{(1-z)^3}{3.2^3} - \ldots$. [3]

4. a) Let f be analytic in a bounded region G and continuous on $\bar{G}$ (closure of G) and $M = \max_{z\in\partial G} |f(z)|$, ∂G being the boundary of G. Prove that $|f(z)| < M$ in G unless f is a constant function. [7]

   b) Let f be a non-constant analytic function on a bounded region D and continuous on $\bar{D}$ and suppose that |f(z)| = constant on ∂D, boundary of D. Prove that f has at least one zero in D. [3]

5. a) If α be a pole of f, prove that $f(z) \to \infty$ as $z \to \alpha$. [4]

   b) Let $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ be a non-zero analytic function. Suppose 0 is a limit point of the zeros of f. Prove that 0 is an essential singularity for f. [4]

   c) Expand $\dfrac{1}{z^2 + 2z - 8}$ in the form of a Laurent series in the annulus $2 < |z+2| < 4$. [2]

6. a) State Rouche's theorem on analytic function. Hence show that all the roots of the equation $z^7 - 5z^3 + 12 = 0$ lie in the annulus bounded by the circles $|z| = 1$ and $|z| = 2$. [1+4]

   b) Let f be a meromorphic function in a region G and γ be a simple closed curve in G such that f is analytic on γ and $f(z) \neq 0$ on γ. Also let N be the number of zeros of f within γ and P be the number of poles of f with in γ, a zero or a pole of order m being counted m times. Prove that $\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = N - P$. [5]

7. a) Evaluate **any one** of the following real integrals by contour integration : [5]

i) $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$

ii) $\int_{0}^{\infty}\frac{\cos ax}{(x^2+b^2)^2}dx, a>0, b>0$

b) If $f(z)=\frac{\cot \pi z}{(z-a)^2}$, $a \neq$ integer, find the residues of f at its singularities. [5]

8. a) Define Cross-ratio of four distinct points in $\mathbb{C}_{\infty}$. Prove that the Cross-ratio of four distinct points remains invariant under a bilinear transformation. [5]

b) Find the Bilinear transformation which maps the points $z=\infty, 0, 1$ into the points w = 0, 1, ∞ respectively and show that this Bilinear transformation maps

i) the real axis $\operatorname{Im} z = 0$ on the real axis $\operatorname{Im} w = 0$.

ii) upper half-plane $\operatorname{Im} z > 0$ on the upper half plane $\operatorname{Im} w > 0$.

iii) lower half-plane $\operatorname{Im} z < 0$ on the lower half-plane $\operatorname{Im} w < 0$. [5]

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